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Virasoro orbits, AdS_3 quantum gravity and entropy*

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Abstract: We analyse the canonical structure of AdS_3 gravity in terms of the coadjoint orbits of the Virasoro group. There is one subset of orbits, associated to BTZ black hole solutions, that can be described by a pair of chiral free fields with a background charge. There is also a second subset of orbits, associated to point-particle solutions, that are described by two pairs of chiral free fields obeying a constraint. All these orbits admit Kähler quantization and generate a Hilbert space which, despite of having $\Delta_0(\bar{\Delta}_0) = 0$, does not provide the right degeneracy to account for the Bekenstein-Hawking entropy due to the breakdown of modular invariance. Therefore, additional degrees of freedom, reestablishing modular invariance, are necessarily required to properly account for the black hole entropy.

Keywords: Black Holes in String Theory, Conformal and W Symmetry, Black Holes.

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Contents

1. Introduction

Three-dimensional quantum gravity with a negative cosmological constant provides an interesting example of the general duality relation proposed in [1, 2, 3] between string theory on anti-de Sitter space (AdS) times a compact space and a conformal field theory (CFT) on the boundary. It was pointed out in [4] that gravity on AdS_3 is a two-dimensional CFT with a classical central charge $c_{cl} = \frac{3}{2}\frac{\ell}{G}$, where G is Newton's constant and $-1/\ell^2$ is the cosmological constant. The physical relevance of 2+1 quantum gravity has recently increased [5] since the near-horizon geometry of black holes arising in string theory can be related to that of the three-dimensional BTZ black holes [6]. Strominger [5] has proposed an unified treatment to account for the Bekenstein-Hawking entropy of all black holes whose near-horizon geometries are locally AdS_3 without using supersymmetry or string theory. This includes the black strings studied in [7] as well as the BTZ black holes. The observation of [5] is based on Cardy's formula [8] for the asymptotic density of states of a unitary and modular invariant two-dimensional CFT with central charge c and eigenvalues $\Delta(\bar{\Delta})$ of $L_0(\bar{L}_0)$

$$S = 2\pi\sqrt{\frac{c\Delta}{6}} + 2\pi\sqrt{\frac{c\bar{\Delta}}{6}}\,. \tag{1.1}$$

As noted in [5] this expression coincides, for $\Delta, \bar{\Delta} \gg c$, with the Bekenstein-Hawking black hole entropy

$$S = \frac{Area}{4G}\,, \tag{1.2}$$

since, for BTZ black holes, one has

$$\begin{aligned} \Delta &= \frac{1}{2}(\ell M + J) + \frac{\ell}{16G} \\ \bar{\Delta} &= \frac{1}{2}(\ell M - J) + \frac{\ell}{16G}\,. \end{aligned} \tag{1.3}$$

The validity of Cardy's formula (1.1) requires that the lowest eigenvalues $(\Delta_0, \bar{\Delta}_0)$ of L_0 and $\bar{L}_0$ vanish, otherwise the asymptotic level density is controlled by the effective central charge $c_{eff} = c - 24\Delta_0$ [9]. This hidden assumption for the Cardy's formula turns out to be very important [10, 11] because it has been argued [12],using the Chern-Simons formulation of the theory [13, 14], that the CFT at spatial infinity for AdS_3 gravity is Liouville theory. However, the analysis of [12] is not complete since the zero modes and the associated holonomies are not considered. Moreover, the lowest eigenvalues $(\Delta_0, \bar{\Delta}_0)$ of L_0 and $\bar{L}_0$ are not zero for normalizable states in Liouville theory [15]

$$\Delta_0 = \frac{c-1}{24} \tag{1.4}$$

and therefore the central charge in (1.1) should indeed be replaced by $c_{eff} = 1$. This implies that the Liouville theory does not have enough states to account for the black hole entropy. Although supersymmetry suggests that the minimum eigenvalue of $L_0(\bar{L}_0)$ vanishes [10], the super-Liouville theory has the same drawback and fails to give the right degeneracy ($c_{eff} = 3/2$). These difficulties were interpreted in [16] suggesting that gravity represents a thermodynamical description of the dual CFT with the Liouville field emerging as a kind of collective coordinate. It has also been argued [17, 18] that string theory on a AdS_3 background could correctly account for the Bekenstein-Hawking entropy. A recent attempt to attack this problem within gravity theory has been proposed in [19, 20] by extending the asymptotic symmetry algebra with new generators.

The aim of this paper is to approach this problem from an analysis of the phase space of the theory in terms of the coadjoint orbits of the Virasoro group [21].

2. Virasoro orbits and gravity on AdS_3

To properly define a gravity theory on AdS_3 we have to provide boundary conditions for the fields at infinity. One can assume that the physical metric field approaches to the AdS_3 metric

$$ds^2 = -\left(\frac{r^2}{\ell^2}+1\right)dt^2 + \left(\frac{r^2}{\ell^2}+1\right)^{-1} dr^2 + r^2 d\theta^2 \,, \tag{2.1}$$

where θ and r are the angular and radial coordinates, as follows

$$g_{+-} = -\frac{r^2}{2} + \gamma_{+-}(x^+, x^-) + \mathcal{O}\left(\frac{1}{r}\right) \tag{2.2}$$

$$g_{\pm\pm} = \gamma_{\pm\pm}(x^+, x^-) + \mathcal{O}\left(\frac{1}{r}\right) \tag{2.3}$$

$$g_{\pm r} = \frac{\gamma_{\pm r}(x^+, x^-)}{r^3} + \mathcal{O}\left(\frac{1}{r^4}\right) \tag{2.4}$$

$$g_{rr} = \frac{\ell^2}{r^2} + \frac{\gamma_{rr}(x^+, x^-)}{r^4} + \mathcal{O}\left(\frac{1}{r^5}\right), \tag{2.5}$$

where $x^{\pm} \equiv t/\ell \pm \theta$. These boundary conditions allow a well defined action of two copies of the Virasoro group through space-time diffeomorphisms. The infinitesimal diffeomorphisms $\zeta^a(r,t,\theta)$ preserving the boundary conditions are

$$\zeta^+ = 2\epsilon^+ + \frac{\ell^2}{r^2}\partial_-^2\epsilon^- + \mathcal{O}\left(\frac{1}{r^4}\right), \tag{2.6}$$

$$\zeta^- = 2\epsilon^- + \frac{\ell^2}{r^2}\partial_+^2\epsilon^+ + \mathcal{O}\left(\frac{1}{r^4}\right), \tag{2.7}$$

$$\zeta^r = -r(\partial_+\epsilon^+ + \partial_-\epsilon^-) + \mathcal{O}\left(\frac{1}{r}\right), \tag{2.8}$$

where the functions $\epsilon^{\pm}$ depend on $x^{\pm}$ ($\epsilon^{\pm}(r,t,\theta) = \epsilon^{\pm}(x^{\pm})$). By using diffeomorphisms which can be regarded as "gauge transformations" ($\epsilon^{\pm} = 0$) one can bring a general metric satisfying the equations of motion

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\frac{1}{\ell^2}g_{\mu\nu} \tag{2.9}$$

into the form [22]

$$ds^2 = \frac{\ell^2}{r^2}dr^2 - r^2dx^+dx^- + \gamma_{++}(dx^+)^2 + \gamma_{--}(dx^-)^2 + \mathcal{O}\left(\frac{1}{r}\right), \tag{2.10}$$

where $\gamma_{\pm}(x^{\pm})$ are chiral functions. If either $\gamma_{++} = 0$ or $\gamma_{--} = 0$, the omitted corrections vanish. Recently [23], it has been obtained an exact general solution, which can be rewritten as

$$ds^2 = \frac{\ell^2}{r^2}dr^2 - \left(rdx^- - \frac{\gamma_{++}}{r}dx^+\right)\left(rdx^+ - \frac{\gamma_{--}}{r}dx^-\right). \tag{2.11}$$

A special class of solutions verifying the boundary conditions (2.2)–(2.5) are the BTZ black holes [6]. They correspond to constant functions for γ_{++}, γ_{--}:

$$\gamma_{\pm\pm} = 2G\ell(M\ell \pm J), \tag{2.12}$$

where M is the black hole mass and J the angular momentum.

The physical excitations can be naturally defined by the action of the "would-be gauge" diffeomorphisms on the topologically inequivalent geometries to AdS_3 (2.1). Obviously, the geometries obtained by a discrete identification of AdS_3 cannot be related by diffeomorphisms and therefore the physical phase space $\mathcal{M}$ of the theory is the collection of the diffeomorphism orbits through the topologically different solutions. We shall now provide a detailed description of this phase space. To this end we have to know the action of the diffeomorphisms on the solutions (2.10).

The action of the infinitesimal diffeomorphisms (2.6)–(2.8) on the functions $\gamma_{\pm\pm}$ is

$$\delta_{\epsilon^{\pm}}\gamma_{\pm\pm} = 2\left(\epsilon^{\pm}\partial_{\pm}\gamma_{\pm\pm} + 2\gamma_{\pm\pm}\partial_{\pm}\epsilon^{\pm}\right) - \ell^2\partial_{\pm}^3\epsilon^{\pm}. \tag{2.13}$$

and the Noether charges $J[\xi]$ associated with them are

$$J[\xi] = \frac{1}{16\ell G} \int d\phi \left\{ \epsilon^+ (4\gamma_{++} + \ell^2) + \epsilon^- (4\gamma_{--} + \ell^2) \right\} , \qquad (2.14)$$

These expressions allow to relate $\gamma_{\pm\pm}$ to the stress tensor $\Theta_{\pm\pm}$ of a conformal field theory on the sphere: $\Theta_{\pm\pm} = \frac{1}{4\ell G}\gamma_{\pm\pm}$. With this identification the transformation law (2.13) for $\Theta_{\pm\pm}$ is

$$\delta_{\epsilon^\pm} \Theta_{\pm\pm} = 2 \left(\epsilon^\pm \partial_\pm \Theta_{\pm\pm} + 2\Theta_{\pm\pm} \partial_\pm \epsilon^\pm \right) - \frac{c}{12} \partial_\pm^3 \epsilon^\pm , \qquad (2.15)$$

where the classical central charge can be worked out immediately

$$c_{cl} = \frac{3}{2} \frac{\ell}{G} . \qquad (2.16)$$

The Fourier components $L_n(\bar{L}_n)$ of $\Theta_{++}(\Theta_{--})$ close down the Virasoro algebra in the Ramond form

$$\begin{aligned} i \left\{ L_n, L_m \right\} &= (n-m) L_{n+m} + \frac{c}{12} n^3 \delta_{n,-m} , \\ i \left\{ \bar{L}_n, \bar{L}_m \right\} &= (n-m) \bar{L}_{n+m} + \frac{c}{12} n^3 \delta_{n,-m} , \\ \left\{ L_n, \bar{L}_m \right\} &= 0 . \end{aligned} \qquad (2.17)$$

The integrated form of (2.13) is

$$\gamma_{\pm\pm} \longrightarrow (\partial_\pm F_\pm)^2 \gamma_{\pm\pm} - \frac{\ell^2}{2} \left(\frac{\partial_\pm^3 F_\pm}{\partial_\pm F_\pm} - \frac{3}{2} \left(\frac{\partial_\pm^2 F_\pm}{\partial_\pm F_\pm} \right)^2 \right) , \qquad (2.18)$$

where $F_\pm(x^\pm)$ are diffeomorphisms of the sphere S^1 parametrized by $e^{ix^\pm}$. The expression (2.18) turns out to be equivalent to the coadjoint action $Ad^*(F_\pm)$ of the Virasoro group [21] and therefore the contribution of (2.18) to the phase space can be identified with some coadjoint orbit of the Virasoro group DiffS^1/H, where H is the stationary subgroup of the orbit. The most interesting orbits [21] emerge when $\gamma_{\pm\pm}$ are constant, and this is the case in our theory (see (2.12)). For a generic constant value $\gamma_{\pm\pm} = 8\pi\ell G b_0^\pm$ the subgroup H is the rotation group S^1 and the coadjoint orbit is then DiffS^1/S^1. However, for special values of $b_0^\pm$

$$b_0^\pm = -n^2 \frac{c_{cl}}{48\pi} , \qquad (2.19)$$

$n = 1, 2, 3, \ldots$, the stationary subgroup becomes larger

$$H = \mathrm{Diff}S^1/\mathrm{SL}^{(n)}(2, \mathbb{R}) , \qquad (2.20)$$

where $SL^{(n)}(2,\mathbb{R})$ is generated by L_0, L_n y L_{-n}. Since the minimum value of $b_0^\pm$ is given by anti-de Sitter space ($b_0^\pm = -c_{cl}/48\pi$) the relevant orbits of our problem are the following

$$b_0^\pm = -\frac{c_{cl}}{48\pi}\,, \qquad \mathrm{Diff}S^1/\mathrm{SL}^{(1)}(2,\mathbb{R}) \oplus \mathrm{Diff}S^1/\mathrm{SL}^{(1)}(2,\mathbb{R}) \tag{2.21}$$

$$b_0^\pm > -\frac{c_{cl}}{48\pi}\,, \qquad \mathrm{Diff}S^1/S^1 \oplus \mathrm{Diff}S^1/S^1\,. \tag{2.22}$$

The sector $b_0^\pm < 0$ corresponds to classical point-particle solutions [24].

These orbits naturally inherit the symplectic two-form [21, 25, 26]

$$\omega = \omega_+ + \omega_-\,, \tag{2.23}$$

where

$$\omega_\pm = \frac{c_{cl}}{48\pi}\delta\int_0^{2\pi} dx^\pm \frac{\delta\partial_\pm^2 F_\pm}{\partial_\pm F_\pm} + b_0^\pm \delta\int_0^{2\pi} dx^\pm \partial_\pm F_\pm \delta F_\pm\,. \tag{2.24}$$

Note that, for convenience, we are still using the group variables to parametrize the orbits.

The Fourier expansion of $F_\pm$

$$F_\pm(x^\pm) = x^\pm + \frac{1}{2\pi}\sum_{k\neq 0} s_k^\pm e^{-ikx^\pm} \tag{2.25}$$

implies that

$$\omega_\pm = -\frac{i}{24}c(k^3 - n^2 k)\delta s_{-k}^\pm \wedge \delta s_k^\pm + \mathcal{O}(s)\,. \tag{2.26}$$

So, to lowest order in a $1/c_{cl}$ expansion the Poisson brackets of $s_k (k\neq 0)$ are

$$\left\{s_k^\pm, s_r^\pm\right\} = i\frac{24}{c}(k^3 - n^2 k)^{-1}\delta_{k,-r} + \mathcal{O}\left(\frac{1}{c_{cl}^2}\right), \tag{2.27}$$

which are similar to the Poisson brackets of free bosons. However, it is well-known that a symplectic structure can always be written, at least locally, in the standard form $\omega = \sum_i \delta p_i \wedge \delta q_i$. In our case, the natural ansatz for the Darboux fields $\phi_\pm$ is

$$\phi_\pm = \sqrt{\frac{c_{cl}}{3}}\left(\frac{1}{2}\ln\partial_\pm F_\pm + \alpha_\pm F_\pm\right), \tag{2.28}$$

where $\alpha_\pm$ are arbitrary real parameters. This gives a symplectic form

$$\begin{aligned}\omega_\pm &= \frac{1}{4\pi}\delta\int_0^{2\pi} dx^\pm \partial_\pm\phi_\pm\delta\phi_\pm \\ &= \frac{c_{cl}}{48\pi}\delta\int_0^{2\pi} dx^\pm\frac{\delta\partial_\pm^2 F_\pm}{\partial_\pm F_\pm} + \frac{\ell\alpha_\pm^2}{8\pi G}\delta\int_0^{2\pi} dx^\pm\partial_\pm F_\pm\delta F_\pm\,.\end{aligned} \tag{2.29}$$

For $b_0^\pm \geq 0$, in order to recover (2.24), we must choose

$$\alpha_\pm = \sqrt{\frac{8\pi G}{\ell}b_0^\pm}\,. \tag{2.30}$$

Then, the stress tensor takes the form of improved chiral free fields

$$\Theta_{\pm\pm} = \frac{1}{2}\left((\partial_\pm\phi_\pm)^2 - \sqrt{\frac{\ell}{2G}}\partial_\pm^2\phi_\pm\right). \tag{2.31}$$

So, the subset of orbits $b_0^\pm \geq 0$ can be described in terms of a pair of chiral fields $\phi_\pm$ whose zero-modes are related to the $b_0^\pm$ parameters through

$$\phi_\pm(x^\pm \pm 2\pi) = \phi_\pm(x^\pm) \pm 2\pi\sqrt{4\pi b_0^\pm}\,. \tag{2.32}$$

We should note that, in case that $b_0^+ = b_0^-$ (i.e. $J = 0$), the left and right moving sectors can be summed up to produce a scalar free field $\phi = \phi_+ + \phi_-$ which, in turn, can be mapped via a Bäcklund transformation into a Liouville field [27, 28]

$$\phi_L = \sqrt{\frac{c_{cl}}{3}}\left(\frac{1}{2}\ln\frac{\partial_+A_+\partial_-A_-}{(1+\frac{\lambda^2}{2}A_+A_-)^2}\right), \tag{2.33}$$

where $A_\pm = F_\pm$ if $\alpha_+ = \alpha_- = 0$ and $A_\pm = \frac{1}{2\alpha_\pm}e^{2\alpha_\pm F_\pm}$ if $\alpha_+ = \alpha_- \neq 0$, and λ^2 is an arbitrary constant. In this situation we have[1]

$$\omega = \omega_+ + \omega_- = \frac{1}{2\pi}\int_0^{2\pi} dx\delta\dot{\phi}_L\delta\phi_L \tag{2.34}$$

and

$$\Theta_{\pm\pm} = \frac{1}{2}\left[(\partial_\pm\phi_L)^2 - \sqrt{\frac{\ell}{2G}}\partial_\pm^2\phi_L\right]. \tag{2.35}$$

This way we recover the results of [12] obtained from the Chern-Simons gauge theory by implementing a Hamiltonian reduction of two chiral WZW models. However, the analysis of [12] does not consider the zero modes and, therefore, cannot see the details of all the Virasoro orbits. In fact, only for $J = 0$ one can construct a Liouville field from the chiral free fields $\phi_\pm$.

The situation for $b_0^\pm < 0$ is more involved. It would be necessary to choose $\alpha_\pm$ imaginary, so that these orbits cannot be described in terms of a pair of real chiral fields $\phi_\pm$ only. In general we have:

$$\Theta_{\pm\pm} = \frac{1}{2}\left\{(\partial_\pm\phi_\pm)^2 - \sqrt{\frac{\ell}{2G}}\partial_\pm^2\phi_\pm\right\} + \left(2\pi(b_0^\pm)^2 - \frac{\ell\alpha_\pm^2}{4G}\right)(\partial_\pm F_\pm)^2 \tag{2.36}$$

Nevertheless, we can describe the subset of orbits $-c_{cl}/48\pi \leq b_0^\pm < 0$ by two pairs of chiral fields $\varphi_\pm$, $\eta_\pm$:

$$\begin{aligned}\sqrt{\frac{3}{c_{cl}}}\varphi_\pm &= \frac{1}{2}\ln\partial_\pm F_\pm \\ \eta_\pm &= F_\pm\,,\end{aligned} \tag{2.37}$$

[1] The Bäcklund transformation defines a proper canonical transformation only if the monodromy of the chiral functions $A_\pm$ of the Liouville field is hyperbolic ($\alpha_+ = \alpha_- \neq 0$) or parabolic ($\alpha_+ = \alpha_- = 0$).

obeying the constraint

$$e^{2\sqrt{3/c}\varphi_\pm} = \partial_\pm \eta_\pm \, . \tag{2.38}$$

The symplectic form is

$$\omega_\pm = \frac{1}{4\pi}\delta \int_0^{2\pi} dx^\pm \partial_\pm \varphi_\pm \delta \varphi_\pm + b_0^\pm \delta \int_0^{2\pi} dx^\pm \partial_\pm \eta_\pm \delta \eta_\pm \tag{2.39}$$

and the stress tensor becomes

$$\Theta_{\pm\pm} = \frac{1}{2}\left[(\partial_\pm \varphi_\pm)^2 - \sqrt{\frac{\ell}{2G}} \partial_\pm^2 \varphi_\pm \right] + 2\pi b_0^\pm (\partial_\pm \eta_\pm)^2 \, . \tag{2.40}$$

In conclusion, the canonical structure of the orbits with $b_0^\pm < 0$ is captured by two pairs of chiral free fields with indefinite signature, one pair with a background charge and the other without improvement, verifying a constraint. We must note that, in contrast with the subset of orbits $b_0^\pm \geq 0$, the presence of the parameters $b_0^\pm$ in the stress tensor (2.40) makes difficult to derive it from a boundary action. This fact will be reflected in the absence of modular invariance in the contribution of these orbits to the Hilbert space.

One can equivalently describe the orbits $-c_{cl}/48\pi \leq b_0^\pm < 0$ in terms of a complex chiral scalar field

$$\phi_\pm = \sqrt{\frac{c_{cl}}{3}} \left(\frac{1}{2} \ln \partial_\pm F_\pm + i|\alpha_\pm| F_\pm \right) , \tag{2.41}$$

which can be rewritten as

$$\phi_\pm = \sqrt{\frac{c_{cl}}{3}} \frac{1}{2} \ln \partial_\pm A_\pm \, , \tag{2.42}$$

where

$$A_\pm = \frac{1}{2i|\alpha_\pm|} e^{2i|\alpha_\pm| F_\pm} \, . \tag{2.43}$$

The fields $A_\pm$ have elliptic monodromy

$$A_\pm(x^\pm \pm 2\pi) = e^{\pm\theta_\pm i} A_\pm(x^\pm) \, , \tag{2.44}$$

where $\theta_\pm = 4\pi|\alpha_\pm| \in (0, 2\pi]$. Note that anti-de Sitter space ($b_0^+ = b_0^- = -c_{cl}/48\pi$) corresponds to the trivial monodromy $\theta_\pm = 2\pi$. For spinless solutions ($J = 0$), i.e. $\theta_+ = \theta_-$, we can again join the two chiral sectors to give a real Liouville field, like in (2.33), with symplectic form and stress tensor given by (2.34) and (2.35).

3. Quantization

We have seen that the subset of the Virasoro orbits with $b_0^\pm \geq 0$ are described by two chiral free fields with a classical background charge $Q_{cl} = \sqrt{c_{cl}/3}$ and this allows a straightforward quantization. At the quantum level

$$Q = \sqrt{\frac{c_{cl}}{3}} + 2\sqrt{\frac{3}{c_{cl}}} \tag{3.1}$$

and the central charge is

$$c = 1 + 3Q^2 \,. \tag{3.2}$$

Due to the presence of the background charge, the state-operator correspondence for momentum eigenstates have the following form

$$e^{ip_\pm\varphi_\pm(0)+\frac{Q}{2}\varphi_\pm(0)} \mid 0\rangle = \mid p_\pm\rangle \tag{3.3}$$

and it follows that

$$\begin{aligned} L_0 \mid p_\pm\rangle &= \frac{Q^2}{8} + \frac{p_+^2}{2} \mid p_+\rangle \\ \bar{L}_0 \mid p_-\rangle &= \frac{Q^2}{8} + \frac{p_-^2}{2} \mid p_-\rangle \,. \end{aligned} \tag{3.4}$$

Therefore the contribution of this sector to the Hilbert space is

$$\bigoplus_{\Delta,\bar{\Delta}\geq\frac{c-1}{24}} \mathcal{H}_\Delta \otimes \mathcal{H}_{\bar{\Delta}} \,, \tag{3.5}$$

where $\mathcal{H}_\Delta(\mathcal{H}_{\bar{\Delta}})$ are Virasoro representations with lowest $L_0(\bar{L}_0)^2$ eigenvalue $\Delta(\bar{\Delta})$ and central charge (3.2).

In contrast, for $-c_{cl}/48\pi \leq b_0^\pm < 0$ the canonical structure of the orbits is more involved. We need two pairs of chiral fields $(\varphi_\pm,\ \eta_\pm)$ obeying the constraint (2.38). The quantization can be carried out consistently imposing the following condition on the physical states $\mid\psi\rangle$

$$\langle\psi \mid : e^{2\sqrt{\frac{3}{c}}\varphi_\pm} : - : \partial_\pm\eta_\pm : \mid \psi\rangle = 0 \,. \tag{3.6}$$

It is important to note that the two terms of the quantum constraint are primary fields with the same conformal weight. The first term is a chiral vertex operator $: e^{2\alpha\phi} :$ with a conformal dimension $\Delta(\bar{\Delta}) = -\frac{1}{2}\alpha^2 + \frac{1}{2}\alpha Q$, due to the presence of the background charge $Q = \sqrt{c_{cl}/3} + 2\sqrt{3/c_{cl}}$. A simple calculation gives $\Delta(\bar{\Delta}) = 1$ and coincides with the dimension of the second term since $\eta_\pm$ are free fields without improvements. Moreover, the quantum Virasoro algebras are generated by the operator version of (2.40), which weakly commute with the constraint giving rise, in the light of the AdS/CFT duality, to the central charge (3.2). Therefore, the quantum central charge of 2+1 gravity coincides with that of Liouville theory with classical central charge $c_{cl} = \frac{3}{2}\frac{\ell}{G}$. However, the full Hilbert space is not isomorphic to that of quantum Liouville theory. The sector coming from the classical point-particle solutions add to the Hilbert space the following Virasoro representations

$$\bigoplus_{\frac{c-1}{24}>\Delta,\bar{\Delta}\geq 0} \mathcal{H}_\Delta \otimes \mathcal{H}_{\bar{\Delta}} \,. \tag{3.7}$$

From the geometrical point of view these results are consistent whit the fact that all orbits with $b_0^\pm \geq -c_{cl}/48\pi$ can be quantized because they posses a Kähler

[2]These are now the usual Neveu-Schwarz $L_0(\bar{L}_0)$ operators.

JHEP05(1999)009

structure [21]. It is interesting to remark that all the orbits DiffS^1/S^1 are Kähler manifolds [29], but only Diff$S^1/\mathrm{SL}^{(n)}(2,\mathbb{R})$ with $n = 1$ admit a Kähler structure [21]. In other words, only AdS_3 ($b_0^{\pm} = -c_{cl}/48\pi$) generate a quantizable orbit with a $\mathrm{SL}(2,\mathbb{R})$ symmetry.

4. Conclusions and final comments

We have shown that the phase space of AdS_3 gravity, with the Brown-Henneaux boundary conditions, can be described in terms of the coadjoint orbits of the Virasoro group and splits into two sectors. The sector associated to classical black hole solutions is described by a pair of chiral free fields with a background charge giving rise to a quantum central charge equal to that of a Liouville theory. However, only when $J = 0$ the two chiral free fields can be summed up to produce a scalar field which can also be mapped, through a canonical transformation, into a Liouville field with hyperbolic and parabolic solutions. The second sector requires two pairs of chiral free fields, one of then with a background charge, obeying a special constraint. Moreover, this sector is canonically equivalent, for $J = 0$, to a classical Liouville field with elliptic monodromy. Nevertheless it is important to point out that, although the classical solutions of three-dimensional gravity with[3] $J = 0$ can be associated to classical solutions of Liouville theory, we have seen that the primary description of the gravity theory appears in terms of chiral free fields and therefore the correspondence with Liouville theory is not valid at the quantum level, as it has been suggested in [16]. Only remains true in the first sector, which can be related to normalizable solutions of quantum Liouville theory [15]. The set of Virasoro representations emerging in this sector (3.5) is modular invariant, but this is no longer true for the second sector and Cardy's formula does not apply. This can be checked immediately because, by direct counting, the asymptotic density of states is the same as in the first sector and it is controlled by $c_{eff} = 1$. Therefore, a Hilbert space of a CFT which could explain the Bekenstein-Hawking entropy requires an enlarged Hilbert space

$$\bigoplus_{\Delta,\bar{\Delta}\geq 0} N_{\Delta,\bar{\Delta}}\mathcal{H}_{\Delta}\otimes\mathcal{H}_{\bar{\Delta}}\,, \tag{4.1}$$

where the positive integer coefficients $N_{\Delta,\bar{\Delta}}$, which stand for the multiplicities of the corresponding Virasoro representations, are such that ensures modular invariance. The gravity theory with the standard Brown-Henneaux boundary conditions is able to see the different Virasoro representations entering the Hilbert space, but not the corresponding multiplicities. This work is left to an additional microscopic structure like string theory [17, 18] or the twisted states recently introduced in [20].

The recent paper [31] also analyzes three-dimensional gravity using the coadjoint orbits of the Virasoro group.

[3] There is a striking analogy between this condition an a similar one emerging in CFT on AdS_3 [30]

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JHEP05(1999)009

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JHEP05(1999)009